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Toroidal solitons in Nicole-type models

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Abstract. A family of modified Nicole models is introduced. We show that for particular members of the family a topological soliton with a non-trivial value of the Hopf index exists. The form of the solitons as well as their energy and topological charge is explicitly found. They appear to be identical to the so-called eikonal knots. The relation between energy and topological charge of the solution is also presented. Quite interestingly it seems to differ drastically from the standard Vakulenko–Kapitansky formula.

1 Introduction

It is widely known that knotted solitons, i.e. topological solutions with a non-trivial value of the Hopf invariant, play a prominent role in the modern physics [1], chemistry [2] and biology [3]. In particular, it has been suggested by Faddeev and Niemi [4] that effective quasi-particles in the low energy limit of the quantum gluodynamics, so-called glueballs, may be described as knotted flux-tubes of the gauge field. In this picture a non-vanishing value of the topological charge provides stability of configurations and, via the Kapitansky–Vakulenko inequality [5] between energy and topological charge, fixes the mass spectrum of glueballs. In fact, they proposed a model (the Faddeev–Niemi model) [4,6], where such knotted solutions have been numerically found [7–9]. It has been also argued by many authors that this model might be derived from the original quantum theory [10–14]. However, up to now, no satisfactory proof that the Faddeev–Niemi model is the low energy limit of the pure quantum Yang–Mill theory has been given.

Unfortunately, due to the fact that the Faddeev–Niemi model belongs to non-exactly solvable theories only numerical [7–9] or some approximated solutions have been obtained [15,16]. In consequence, many problems concerning the properties of the Faddeev–Niemi hopfions have not been solved yet. Therefore, a few models based on the same degrees of freedom and topology but possessing analytical solutions have been constructed [17,18]. For example in the Aratyn–Ferreira–Zimerman model [18] infinitely many solitons with an arbitrary Hopf number have been found. Such toy models gave a chance to understand the connections between topological charge and shape of a solution as well as allowed us to check the energy–charge inequality. On the other hand, in the case of the second widely investigated toy model i.e. the Nicole model [17] (it is the oldest model with explicitly found hopfion) the spectrum of the solutions is scarcely known. Only the simplest hopfion with $|Q_H| = 1$ has been found.

The main aim of the present paper is to prove that slightly modified Nicole models possess in their spectrum of solutions hopfions with topological charge $Q_H = -m^2$, where $m \in Z$, and analyze their properties like shape, energy etc. In particular, we are interested in checking of validity of the Vakulenko–Kapitansky formula.

2 Model and solutions

Let us start with the following Lagrangian density:

$$L = \frac{1}{2}\sigma(\boldsymbol{n})(\partial_\mu \boldsymbol{n}\partial^\mu \boldsymbol{n})^{\frac{3}{2}}, \tag{1}$$

where $\boldsymbol{n}$ is a unit, three component vector field living in the $(3+1)$ Minkowski space-time. This model differs from the original Nicole Lagrangian only via a function σ, which in the case of the Nicole model is just a constant. One can see that the appearance of a non-trivial σ function will result in the explicit breaking of the global $O(3)$ symmetry. Models with this property have been recently with versatility investigated [19,12,20,21]. The physical importance of such models follows from the observation that the Faddeev–Niemi model possesses two massless Goldstone bosons since the spontaneous $O(3)$ symmetry breaking occurs. Thus, in order to get rid of these massless states one is forced to implement the explicit symmetry breaking i.e. to add a new term into the Lagrangian which is not invariant under this global symmetry. Indeed, it has been shown that in some particular patterns of the symmetry breaking the Goldstone bosons are removed from the spectrum of the theory and a mass gap appears [22].

In our work the symmetry breaking function is assumed in the following form:

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$$\sigma(\boldsymbol{n}) = \left(\frac{1+n^3}{1-n^3}\right)^{\frac{3}{2}\left(\frac{1}{m}-1\right)} \left[\frac{1+\frac{1+n^3}{1-n^3}}{1+\left(\frac{1+n^3}{1-n^3}\right)^{\frac{1}{m}}}\right]^3, \tag{2}$$

where m is an integer and positive number. Now, we take advantage of the stereographic projection

$$\boldsymbol{n} = \frac{1}{1+|u|^2}(u+u^*, -\mathrm{i}(u-u^*), |u|^2-1) \tag{3}$$

and rewrite the Lagrangian (1) as follows:

$$L = \left(\frac{|u|^{\frac{1}{m}-1}}{1+|u|^{\frac{2}{m}}}\right)^3 (\partial_\mu u \partial^\mu u^*)^{\frac{3}{2}}. \tag{4}$$

Thus, the pertinent equation of motion reads

$$\frac{3}{2}\partial_\mu \left[\left(\frac{|u|^{\frac{1}{m}-1}}{1+|u|^{\frac{2}{m}}}\right)^3 (\partial_\nu u \partial^\nu u^*)^{\frac{1}{2}} \partial^\mu u\right] - (\partial_\nu u \partial^\nu u^*)^{\frac{3}{2}} \frac{\partial}{\partial u^*}\left[\left(\frac{|u|^{\frac{1}{m}-1}}{1+|u|^{\frac{2}{m}}}\right)^3\right] = 0. \tag{5}$$

Analogously as in the case of the standard Nicole Lagrangian, our model possesses an integrable submodel defined by the additional condition which is nothing else but the eikonal equation [23,24]

$$\partial_\mu u \partial^\mu u = 0. \tag{6}$$

Then, one can adopt the procedure introduced in [25] and construct an infinite family of conserved currents.

One has to remember that solutions of the integrable submodel must obey, except the integrability condition introduced above, also dynamical equations achieved from (5):

$$\partial_\mu \left[\frac{|u|^{\frac{1}{m}-1}}{1+|u|^{\frac{2}{m}}} (\partial_\nu u \partial^\nu u^*)^{\frac{1}{2}} \partial^\mu u\right] = 0. \tag{7}$$

Let us now find topological solutions of the integrable submodel (6) and (7). The first step is to introduce the toroidal coordinates

$$x = \frac{\tilde{a}}{q}\sinh\eta\cos\phi,$$
$$y = \frac{\tilde{a}}{q}\sinh\eta\sin\phi,$$
$$z = \frac{\tilde{a}}{q}\sin\xi, \tag{8}$$

where $q = \cosh\eta - \cos\xi$ and $\tilde{a}$ is a dimensional constant which fixes the scale in the coordinates. Moreover, the solution is assumed to have the following form [18]:

$$u(\eta,\xi,\phi) = f(\eta)\mathrm{e}^{\mathrm{i}m(\xi+\phi)}, \tag{9}$$

where the unknown function f is yet to be determined. It is a well-known fact [18,23,20] that for smooth functions f such that $f(0)=\infty$ and $f(\infty)=0$ the map (9) corresponds to a non-vanishing value of the topological charge. Indeed, one can get that

$$Q_H = -m^2. \tag{10}$$

Inserting Ansatz (9) into (7) we obtain

$$\partial_\eta \left[\frac{f^{\frac{1-m}{m}}}{1+f^{\frac{2}{m}}}\left(f_\eta'^2 + m^2\frac{\cosh^2\eta}{\sinh^2\eta}f^2\right)^{\frac{1}{2}} f_\eta'\right] - m^2\frac{\cosh^2\eta}{\sinh^2\eta}\frac{f^{\frac{1-m}{m}}}{1+f^{\frac{2}{m}}} f\left(f_\eta'^2 + m^2\frac{\cosh^2\eta}{\sinh^2\eta}f^2\right)^{\frac{1}{2}} + \frac{f^{\frac{1-m}{m}}}{1+f^{\frac{2}{m}}}\left(f_\eta'^2 + m^2\frac{\cosh^2\eta}{\sinh^2\eta}f^2\right)^{\frac{1}{2}} f_\eta'\frac{\cosh\eta}{\sinh\eta} = 0. \tag{11}$$

After some calculations, it can be reduced to the more compact form

$$\partial_\mu \ln\left[\frac{f^{\frac{1-m}{m}}}{1+f^{\frac{2}{m}}}\left(f_\eta'^2 + m^2\frac{\cosh^2\eta}{\sinh^2\eta}f^2\right)^{\frac{1}{2}} |f_\eta'|\right] + m^2\frac{\cosh^2\eta}{\sinh^2\eta}\frac{f}{|f'|} + \frac{\cosh\eta}{\sinh\eta} = 0. \tag{12}$$

On the other hand, our submodel is defined not only by the dynamical field equation (7) but also by the constraint (6), which in the case of the upper introduced Ansatz takes the following form:

$$f_\eta'^2 = m^2\frac{\cosh^2\eta}{\sinh^2\eta}f^2. \tag{13}$$

Thus, the dynamical equation can be simplified:

$$\partial_\eta \ln\left[\frac{f^{\frac{1-m}{m}}}{1+f^{\frac{2}{m}}}f'^2\right] = -(m+1)\partial_\eta \ln\sinh\eta. \tag{14}$$

Applying once again the constraint (13) we find that

$$\partial_\eta \ln\left[\frac{\cosh^2\eta}{\sinh^2\eta}\frac{f^{\frac{1+m}{m}}}{1+f^{\frac{2}{m}}}\right] = -(m+1)\partial_\eta \ln\sinh\eta. \tag{15}$$

This differential equation can be easily solved and in consequence we derive an algebraic equation for f:

$$\frac{\cosh^2\eta}{\sinh^2\eta}\frac{f^{\frac{1+m}{m}}}{1+f^{\frac{2}{m}}} = \left(\frac{1}{\sinh\eta}\right)^{m+1}. \tag{16}$$

The solution of this equation reads

$$f(\eta) = \frac{1}{\sinh^m\eta}. \tag{17}$$

One can check that it solves our constraint (13) as well. Thus, the Ansatz (9) where the shape function, i.e. the function f, takes the above obtained form (17) is a static, topologically non-trivial solution of the integrable submodels. One can immediately notice that for $m = 1$ the well-known unit charge hopfion, which is a solution to the standard Nicole model, is reproduced.

It should be underlined that every exact solution (17), labeled by $m \in N$, refers to a different modified Nicole Lagrangian. We have proved that any model (1) with $m \in N$ possesses a topological solution with $Q_H = -m^2$. This is unlikely for the Aratyn–Ferreira–Zimerman model where an infinite family of hopfions has been found. Nonetheless, our calculation shows that also in the framework of the modified Nicole models some exact hopfions with higher than one topological charge can be constructed.

Let us now compute the corresponding energy. Using the stereographic projection we derive

$$E = \int \mathrm{d}^3x \left(\frac{|u|^{\frac{1}{m}-1}}{1+|u|^{\frac{2}{m}}} \right)^3 (\partial_i u \partial_i u^*)^{\frac{3}{2}}. \qquad (18)$$

Taking into account the form of the Ansatz one can rewrite this expression as follows:

$$E = (2\pi)^2 \qquad (19)$$
$$\times \int_0^\infty \mathrm{d}\eta \sinh\eta \left(\frac{f^{\frac{1-m}{m}}}{1+f^{\frac{2}{m}}} \right)^3 \left(f_\eta'^2 + m^2 \frac{\cosh^2\eta}{\sinh^2\eta} f^2 \right)^{\frac{3}{2}}.$$

Finally, inserting our solution we obtain that

$$E = (2\pi)^2 2^{\frac{3}{2}} m^3 \int_0^\infty \frac{\sinh\eta}{\cosh^3\eta} = \sqrt{2}(2\pi)^2 m^3. \qquad (20)$$

Quite interestingly, the energy of the hopfion is related with its topological charge by the following relation:

$$E = \sqrt{2}(2\pi)^2 |Q_H|^{\frac{3}{2}}, \qquad (21)$$

which differs considerably from the standard Vakulenko–Kapitansky formula. Vakulenko and Kapitansky proved that in the case of the Faddeev–Niemi model the energy of the solution is bounded from below by the corresponding topological charge. Namely,

$$E \geq C|Q_H|^{\frac{3}{4}}, \qquad (22)$$

where C is a numerical constant. Recently, new results concerning the upper bound have been presented in [26]. It has been also shown that asymptotically for a large topological charge the energy is proportional to $|Q|^{3/4}$. Moreover, it was checked by direct calculations that this relation is valid for all known solutions of the Aratyn–Ferreira–Zimerman model [18] as well as its generalizations [20,27]. Indeed, the energy grows proportional to $|Q|^{3/4}$.

Here, for the modified Nicole models, such a sublinear behavior is not longer held. Indeed, the exponent characterizing the dependence on the topological index is bigger than one and reads $\frac{3}{2}$. Of course, the Vakulenko–Kapitansky inequality is valid since $E \geq C|Q|^{3/2} \geq C|Q|^{3/4}$. Nonetheless, the different value of the exponent can result in a modification of the interaction between the hopfions. Instead of the standard clustering phenomena (a separated multi-soliton configuration tends to form a clustered, really knotted state) one should rather expect splitting i.e. decay of a soliton with high topological charges into unknots with unit Hopf index.

One can notice that there may be a trivial solution to this unexpected relation between energy and the topological index. Namely, the presented solitons are not the energy minima in the fixed topological sector. Then there may exist less energy solutions which would saturate the Vakulenko–Kapitansky formula. Such a possibility is also interesting as it suggests that stable configurations could be given not by the unknots obtained here but by really knotted solitons. Due to the fact that such a property is observed in the Faddeev–Niemi model [8,9], it would indicate that the Nicole toy model is much more relevant to the investigation of hopfions than the Aratyn–Ferreira–Zimerman model.

However, it must be stressed once again that we do not know whether all hopfions corresponding to any of the modified models follow the relation (21) since only one hopfion for each modified model has been obtained. Thus, as far as no solutions with other values of the topological charge will be found, our energy–charge relation has to be treated only as a conjecture.

3 Conclusions

In the present work, a modification of the Nicole Lagrangian has been considered. For each member of the family of the modified models (1) (labeled by an integer and positive parameter m) a topological solution with $Q_H = -m^2$ has been found. Let us briefly summarize the obtained results.

First of all, we have shown that all solitons are unknots, that is, surfaces corresponding to constant values of the unit, vector field $\boldsymbol{n}$ are toruses. It resembles the situation known from the Aratyn–Ferreira–Zimerman model. This fact can be treated as a disadvantage of the toy models since Faddeev–Niemi hopfions are really knotted objects without toroidal symmetry.

However, the most important observation we have made concerns the energy–charge inequality. As we have discussed before, there are some arguments indicating that the energy of the hopfions for each of the modified Nicole models is proportional to $Q_H^{3/2}$ rather than $Q_H^{3/4}$ as one could expected from the Vakulenko–Kapitansky inequality.

Undoubtedly, further studies are needed. For example, the validity of this conjecture should be checked. We would like to address this problem in a forthcoming paper.

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